ENERGY FUNCTIONALS FOR THE PARABOLIC MONGE-AMPÈRE EQUATION

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1. Introduction

Because of its close connection with the Kähler-Ricci flow, the parabolic complex Monge-Ampère equation on complex manifolds has been studied by many authors. See, for instance, [Cao85, CT02, PS06]. On the other hand, theories for complex Monge-Ampère equation on both bounded domains and complex manifolds were developed in [BT76, Yau78, CKNS85, Koł98]. In this paper, we are going to study the parabolic complex Monge-Ampère equation over a bounded domain.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Denote $\mathcal{Q}_T = \Omega \times (0, T)$ with $T > 0$, $B = \Omega \times \{0\}$, $\Gamma = \partial\Omega \times \{0\}$ and $\Sigma_T = \partial\Omega \times (0, T)$. Let $\partial_p \mathcal{Q}_T$ be the parabolic boundary of $\mathcal{Q}_T$, i.e. $\partial_p \mathcal{Q}_T = B \cup \Gamma \cup \Sigma_T$. Consider the following boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \log\det\left(u_{\alpha\bar\beta}\right) = f(t, z, u) & \text{in } \mathcal{Q}_T, \\ u = \varphi & \text{on } \partial_p \mathcal{Q}_T. \end{cases} \tag{1}$$

where $f \in \mathcal{C}^\infty(\mathbb{R} \times \bar\Omega \times \mathbb{R})$ and $\varphi \in \mathcal{C}^\infty(\partial_p \mathcal{Q}_T)$. We will always assume that

$$\frac{\partial f}{\partial u} \le 0. \tag{2}$$

Then we will prove that

Theorem 1. *Suppose there exists a spatial plurisubharmonic (psh) function $\underline{u} \in \mathcal{C}^2(\bar{\mathcal{Q}}_T)$ such that*

$$\begin{aligned} &\underline{u}_t - \log\det\left(\underline{u}_{\alpha\bar\beta}\right) \le f(t, z, \underline{u}) && \textit{in } \mathcal{Q}_T, \\ &\underline{u} \le \varphi \quad \textit{on } B \qquad \textit{and} \qquad \underline{u} = \varphi && \textit{on } \Sigma_T \cap \Gamma. \end{aligned} \tag{3}$$

Then there exists a spatial psh solution $u \in \mathcal{C}^\infty(\bar{\mathcal{Q}}_T)$ of (1) with $u \ge \underline{u}$ if following compatibility condition is satisfied: $\forall\, z \in \partial\Omega$,

$$\begin{aligned} \varphi_t - \log\det\left(\varphi_{\alpha\bar\beta}\right) &= f(0, z, \varphi(z)), \\ \varphi_{tt} - \left(\log\det(\varphi_{\alpha\bar\beta})\right)_t &= f_t(0, z, \varphi(z)) + f_u(0, z, \varphi(z))\varphi_t. \end{aligned} \tag{4}$$

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Motivated by the energy functionals in the study of the Kähler-Ricci flow, we introduce certain energy functionals to the complex Monge-Ampère equation over a bounded domain. Given $\varphi \in \mathcal{C}^\infty(\partial\Omega)$, denote

$$\mathcal{P}(\Omega, \varphi) = \left\{ u \in \mathcal{C}^2(\bar{\Omega}) \mid u \text{ is psh, and } u = \varphi \text{ on } \partial\Omega \right\}, \tag{5}$$

then define the F^0 functional by following variation formula:

$$\delta F^0(u) = \int_\Omega \delta u \det\left(u_{\alpha\bar{\beta}}\right). \tag{6}$$

We shall show that the F^0 functional is well-defined. Using this F^0 functional and following the ideas of [PS06], we prove that

Theorem 2. *Assume that both φ and f are independent of t, and*

$$f_u \le 0 \qquad \text{and} \qquad f_{uu} \le 0. \tag{7}$$

Then the solution u of (1) exists for $T = +\infty$, and as t approaches $+\infty$, $u(\cdot, t)$ approaches the unique solution of the Dirichlet problem

$$\begin{cases} \det\left(v_{\alpha\bar{\beta}}\right) = e^{-f(z,v)} & \text{in } \mathcal{Q}_T, \\ v = \varphi & \text{on } \partial_p \mathcal{Q}_T, \end{cases} \tag{8}$$

in $\mathcal{C}^{1,\alpha}(\bar{\Omega})$ for any $0 < \alpha < 1$.

Remark: Similar energy functionals have been studied in [Bak83, Tso90, Wan94, TW97, TW98] for the real Monge-Ampère equation and the real Hessian equation with homogeneous boundary condition $\varphi = 0$, and the convergence for the solution of the real Hessian equation was also proved in [TW98]. Our construction of the energy functionals and the proof of the convergence also work for these cases, and thus we also obtain an independent proof of these results. Li [Li04] and Blocki [Bło05] studied the Dirichlet problems for the complex k-Hessian equations over bounded complex domains. Similar energy functional can also be constructed for the parabolic complex k-Hessian equations and be used for the proof of the convergence.

2. A priori $\mathcal{C}^2$ estimate

By the work of Krylov [Kry83], Evans [Eva82], Caffarelli etc. [CKNS85] and Guan [Gua98], it is well known that in order to prove the existence and smoothness of (1), we only need to establish the a priori $\mathcal{C}^{2,1}(\bar{\mathcal{Q}}_T)$[1] estimate, i.e. for solution $u \in \mathcal{C}^{4,1}(\bar{\mathcal{Q}}_T)$ of (1) with

$$u = \underline{u} \quad \text{on} \quad \Sigma_T \cup \Gamma \qquad \text{and} \qquad u \ge \underline{u} \quad \text{in} \quad \mathcal{Q}_T, \tag{9}$$

then

$$\|u\|_{\mathcal{C}^{2,1}(\mathcal{Q}_T)} \le M_2, \tag{10}$$

where M_2 only depends on $\mathcal{Q}_T, \underline{u}, f$ and $\|u(\cdot,0)\|_{\mathcal{C}^2(\bar{\Omega})}$.

[1] $\mathcal{C}^{m,n}(\mathcal{Q}_T)$ means m times and n times differentiable in space direction and time direction respectively, same for $\mathcal{C}^{m,n}$-norm.

Proof of (10). Since u is spatial psh and $u \geq \underline{u}$, so

$$\underline{u} \leq u \leq \sup_{\Sigma_T} \underline{u}$$

i.e.

$$\|u\|_{\mathcal{C}^0(\mathcal{Q}_T)} \leq M_0. \tag{11}$$

Step 1. $|u_t| \leq C_1$ in $\bar{\mathcal{Q}}_T$.

Let $G = u_t(2M_0 - u)^{-1}$. If G attains its minimum on $\bar{\mathcal{Q}}_T$ at the parabolic boundary, then $u_t \geq -C_1$ where C_1 depends on M_0 and $\underline{u}_t$ on Σ. Otherwise, at the point where G attains the minimum,

$$\begin{aligned} G_t \leq 0 \quad &\text{i.e.} \quad u_{tt} + (2M_0 - u)^{-1} u_t^2 \leq 0, \\ G_\alpha = 0 \quad &\text{i.e.} \quad u_{t\alpha} + (2M_0 - u)^{-1} u_t u_\alpha = 0, \\ G_{\bar{\beta}} = 0 \quad &\text{i.e.} \quad u_{t\bar{\beta}} + (2M_0 - u)^{-1} u_t u_{\bar{\beta}} = 0, \end{aligned} \tag{12}$$

and the matrix $G_{\alpha\bar{\beta}}$ is non-negative, i.e.

$$u_{t\alpha\bar{\beta}} + (2M_0 - u)^{-1} u_t u_{\alpha\bar{\beta}} \geq 0. \tag{13}$$

Hence

$$0 \leq u^{\alpha\bar{\beta}}\big(u_{t\alpha\bar{\beta}} + (2M_0 - u)^{-1} u_t u_{\alpha\bar{\beta}}\big) = u^{\alpha\bar{\beta}} u_{t\alpha\bar{\beta}} + n(2M_0 - u)^{-1} u_t, \tag{14}$$

where $(u^{\alpha\bar{\beta}})$ is the inverse matrix for $(u_{\alpha\bar{\beta}})$, i.e.

$$u^{\alpha\bar{\beta}} u_{\gamma\bar{\beta}} = \delta^\alpha{}_\gamma.$$

Differentiating (1) in t, we get

$$u_{tt} - u^{\alpha\bar{\beta}} u_{t\alpha\bar{\beta}} = f_t + f_u\, u_t, \tag{15}$$

so

$$\begin{aligned} (2M_0 - u)^{-1} u_t^2 &\leq -u_{tt} \\ &= -u^{\alpha\bar{\beta}} u_{t\alpha\bar{\beta}} - f_t - f_u\, u_t \\ &\leq n(2M_0 - u)^{-1} u_t - f_u\, u_t - f_t, \end{aligned}$$

hence

$$u_t^2 - (n - (2M_0 - u) f_u) u_t + f_t(2M_0 - u) \leq 0.$$

Therefore at point p, we get

$$u_t \geq -C_1 \tag{16}$$

where C_1 depends on M_0 and f.

Similarly, by considering the function $u_t(2M_0 + u)^{-1}$ we can show that

$$u_t \leq C_1. \tag{17}$$

Step 2. $|\nabla u| \leq M_1$

Extend $\underline{u}|_\Sigma$ to a spatial harmonic function h, then

$$\underline{u} \leq u \leq h \quad \text{in} \quad \mathcal{Q}_T \qquad \text{and} \qquad \underline{u} = u = h \quad \text{on} \quad \Sigma_T. \tag{18}$$

So

$$|\nabla u|_{\Sigma_T} \leq M_1. \tag{19}$$

Let L be the linear differential operator defined by

$$Lv = \frac{\partial v}{\partial t} - u^{\alpha\bar{\beta}} v_{\alpha\bar{\beta}} - f_u v. \tag{20}$$

Then

$$\begin{aligned} L(\nabla u + e^{\lambda |z|^2}) &= L(\nabla u) + L e^{\lambda |z|^2} \\ &\leq \nabla f - e^{\lambda|z|^2}\big(\lambda \sum u^{\alpha\bar{\alpha}} - f_u\big). \end{aligned} \tag{21}$$

Noticed that and both u and $\dot{u}$ are bounded and

$$\det\big(u_{\alpha\bar{\beta}}\big) = e^{\dot{u} - f},$$

so

$$0 < c_0 \leq \det\big(u_{\alpha\bar{\beta}}\big) \leq c_1, \tag{22}$$

where c_0 and c_1 depends on M_0 and f. Therefore

$$\sum u^{\alpha\bar{\alpha}} \geq n c_1^{-1/n}. \tag{23}$$

Hence after taking λ large enough, we can get

$$L(\nabla u + e^{\lambda|z|^2}) \leq 0,$$

thus

$$|\nabla u| \leq \sup_{\partial_p \mathcal{Q}_T} |\nabla u| + C_2 \leq M_1. \tag{24}$$

Step 3. $|\nabla^2 u| \leq M_2$ on Σ.

At point $(p,t) \in \Sigma$, we choose coordinates $z_1, \cdots, z_n$ for Ω, such that at $z_1 = \cdots = z_n = 0$ at p and the positive x_n axis is the interior normal direction of $\partial\Omega$ at p. We set $s_1 = y_1, s_2 = x_1, \cdots, s_{2n-1} = y_n, s_{2n} = x_n$ and $s' = (s_1, \cdots, s_{2n-1})$. We also assume that near p, $\partial\Omega$ is represented as a graph

$$x_n = \rho(s') = \frac{1}{2} \sum_{j,k<2n} B_{jk} s_j s_k + O(|s'|^3). \tag{25}$$

Since $(u - \underline{u})(s', \rho(s'), t) = 0$, we have for $j, k < 2n$,

$$(u - \underline{u})_{s_j s_k}(p,t) = -(u - \underline{u})_{x_n}(p,t) B_{jk}, \tag{26}$$

hence

$$|u_{s_j s_k}(p,t)| \leq C_3, \tag{27}$$

where C_3 depends on $\partial\Omega, \underline{u}$ and M_1.

We will follow the construction of barrier function by Guan [Gua98] to estimate $|u_{x_n s_j}|$. For $\delta > 0$, denote $\mathcal{Q}_\delta(p,t) = \big(\Omega \cap B_\delta(p)\big) \times (0,t)$.

Lemma 3. *Define the function*

$$d(z) = \mathrm{dist}(z, \partial\Omega) \tag{28}$$

and

$$v = (u - \underline{u}) + a(h - \underline{u}) - Nd^2. \tag{29}$$

Then for N sufficiently large and a, δ sufficiently small,

$$\begin{aligned} Lv &\geq \epsilon(1 + \sum u^{\alpha\bar{\alpha}}) && \text{in } \mathcal{Q}_\delta(p,t) \\ v &\geq 0 && \text{on } \partial(B_\delta(p) \cap \Omega) \times (0,t) \\ v(z,0) &\geq c_3|z| && \text{for } z \in B_\delta(p) \cap \Omega \end{aligned} \tag{30}$$

where ϵ depends on the uniform lower bound of he eigenvalues of $\{\underline{u}_{\alpha\bar{\beta}}\}$.

Proof. See the proof of Lemma 2.1 in [Gua98]. ∎

For $j < 2n$, consider the operator

$$T_j = \frac{\partial}{\partial s_j} + \rho_{s_j} \frac{\partial}{\partial x_n}.$$

Then

$$\begin{aligned} T_j(u - \underline{u}) &= 0 && \text{on } \big(\partial\Omega \cap B_\delta(p)\big) \times (0,t) \\ |T_j(u - \underline{u})| &\leq M_1 && \text{on } \big(\Omega \cap \partial B_\delta(p)\big) \times (0,t) \\ |T_j(u - \underline{u})(z,0)| &\leq C_4|z| && \text{for } z \in B_\delta(p) \end{aligned} \tag{31}$$

So by Lemma 3 we may choose C_5 independent of u, and $A >> B >> 1$ so that

$$\begin{aligned} L\big(Av + B|z|^2 - C_5(u_{y_n} - \underline{u}_{y_n})^2 \pm T_j(u - \underline{u})\big) &\geq 0 \quad \text{in } \mathcal{Q}_\delta(p,t), \\ Av + B|z|^2 - C_5(u_{y_n} - \underline{u}_{y_n})^2 \pm T_j(u - \underline{u}) &\geq 0 \quad \text{on } \partial_p \mathcal{Q}_\delta(p,t). \end{aligned} \tag{32}$$

Hence by the comparison principle,

$$Av + B|z|^2 - C_5(u_{y_n} - \underline{u}_{y_n})^2 \pm T_j(u - \underline{u}) \geq 0 \qquad \text{in } \mathcal{Q}_\delta(p,t),$$

and at (p,t)

$$|u_{x_n y_j}| \leq M_2. \tag{33}$$

To estimate $|u_{x_n x_n}|$, we will follow the simplification in [Tru95]. For $(p,t) \in \Sigma$, define

$$\lambda(p,t) = \min\{u_{\xi\bar{\xi}} \mid \text{ complex vector } \xi \in T_p\partial\Omega, \text{ and } |\xi| = 1\}$$

Claim $\lambda(p,t) \geq c_4 > 0$ where c_4 is independent of u.

Let us assume that $\lambda(p,t)$ attains the minimum at (z_0, t_0) with $\xi \in T_{z_o}\partial\Omega$. We may assume that

$$\lambda(z_0, t_0) < \frac{1}{2}\underline{u}_{\xi\bar{\xi}}(z_0, t_0).$$

Take a unitary frame $e_1, \cdots, e_n$ around z_0, such that $e_1(z_0) = \xi$, and $\mathrm{Re}\, e_n = \gamma$ is the interior normal of $\partial\Omega$ along $\partial\Omega$. Let r be the function which defines Ω, then

$$(u - \underline{u})_{1\bar{1}}(z,t) = -r_{1\bar{1}}(z)(u - \underline{u})_\gamma(z,t) \qquad z \in \partial\Omega$$

Since $u_{1\bar{1}}(z_0,t_0) < \underline{u}_{1\bar{1}}(z_0,t_0)/2$, so

$$-r_{1\bar{1}}(z_0)(u - \underline{u})_\gamma(z_0,t_0) \le -\frac{1}{2}\underline{u}_{1\bar{1}}(z_0,t_0).$$

Hence

$$r_{1\bar{1}}(z_0)(u - \underline{u})_\gamma(z_0,t) \ge \frac{1}{2}\underline{u}_{1\bar{1}}(z_0,t) \ge c_5 > 0.$$

Since both ∇u and $\nabla \underline{u}$ are bounded, we get

$$r_{1\bar{1}}(z_0) \ge c_6 > 0,$$

and for δ sufficiently small (depends on $r_{1\bar{1}}$) and $z \in B_\delta(z_0) \cap \Omega$,

$$r_{1\bar{1}}(z) \ge \frac{c_6}{2}.$$

So by $u_{1\bar{1}}(z,t) \ge u_{1\bar{1}}(z_0,t_0)$, we get

$$\underline{u}_{1\bar{1}}(z,t) - r_{1\bar{1}}(z)(u - \underline{u})_\gamma(z,t) \ge \underline{u}_{1\bar{1}}(z_0,t_0) - r_{1\bar{1}}(z_0)(u - \underline{u})_\gamma(z_0,t_0).$$

Hence if we let

$$\Psi(z,t) = \frac{1}{r_{1\bar{1}}(z)}\big(r_{1\bar{1}}(z_0)(u - \underline{u})_\gamma(z_0,t_0) + \underline{u}_{1\bar{1}}(z,t) - \underline{u}_{1\bar{1}}(z_0,t_0)\big)$$

then

$$\begin{aligned}(u - \underline{u})_\gamma(z,t) &\le \Psi(z,t) \qquad \text{on } \big(\partial\Omega \cap B_\delta(z_0)\big) \times (0,T)\\ (u - \underline{u})_\gamma(z_0,t_0) &= \Psi(z_0,t_0).\end{aligned}$$

Now take the coordinate system $z_1, \cdots, z_n$ as before. Then

$$\begin{aligned}(u - \underline{u})_{x_n}(z,t) &\le \frac{1}{\gamma_n(z)}\Psi(z,t) \qquad \text{on } \big(\partial\Omega \cap B_\delta(z_0)\big) \times (0,T)\\ (u - \underline{u})_{x_n}(z_0,t_0) &= \frac{1}{\gamma_n(z_0)}\Psi(z_0,t_0).\end{aligned} \tag{34}$$

where γ_n depends on $\partial\Omega$. After taking C_6 independent of u and $A >> B >> 1$, we get

$$L\Big(Av + B|z|^2 - C_6(u_{y_n} - \underline{u}_{y_n})^2 + \frac{\Psi(z,t)}{\gamma_n(z)} - T_j(u - \underline{u})\Big) \ge 0 \quad \text{in } \mathcal{Q}_\delta(p,t),$$

$$Av + B|z|^2 - C_6(u_{y_n} - \underline{u}_{y_n})^2 + \frac{\Psi(z,t)}{\gamma_n(z)} - T_j(u - \underline{u}) \ge 0 \quad \text{on } \partial_p\mathcal{Q}_\delta(p,t).$$

So

$$Av + B|z|^2 - C_6(u_{y_n} - \underline{u}_{y_n})^2 + \frac{\Psi(z,t)}{\gamma_n(z)} - T_j(u - \underline{u}) \ge 0 \qquad \text{in } \mathcal{Q}_\delta(p,t),$$

and

$$|u_{x_n x_n}(z_0,t_0)| \le C_7.$$

Therefore at (z_0, t_0), $u_{\alpha\bar{\beta}}$ is uniformly bounded, hence

$$u_{1\bar{1}}(z_0, t_0) \geq c_4$$

with c_4 independent of u. Finally, from the equation

$$\det u_{\alpha\bar{\beta}} = e^{\dot{u}-f}$$

we get

$$|u_{x_n x_n}| \leq M_2.$$

Step 4. $|\nabla^2 u| \leq M_2$ in $\mathcal{Q}$.

By the concavity of $\log\det$, we have

$$L(\nabla^2 u + e^{\lambda|z|^2}) \leq O(1) - e^{\lambda|z|^2}\big(\lambda \sum u^{\alpha\bar{\alpha}} - f_u\big)$$

So for λ large enough,

$$L(\nabla^2 u + e^{\lambda|z|^2}) \leq 0,$$

and

$$\sup |\nabla^2 u| \leq \sup_{\partial_p \mathcal{Q}_T} |\nabla^2 u| + C_8 \tag{35}$$

with C_8 depends on M_0, Ω and f.

□

3. The Functionals I, J and F^0

Let us recall the definition of $\mathcal{P}(\Omega, \varphi)$ in (5),

$$\mathcal{P}(\Omega, \varphi) = \left\{u \in \mathcal{C}^2(\bar{\Omega} \mid u \text{ is psh, and } u = \varphi \text{ on } \partial\Omega\right\}.$$

Fixing $v \in \mathcal{P}$, for $u \in \mathcal{P}$, define

$$I_v(u) = -\int_\Omega (u - v)(\sqrt{-1}\partial\bar{\partial}u)^n. \tag{36}$$

Proposition 4. *There is a unique and well defined functional J_v on $\mathcal{P}(\Omega, \varphi)$, such that*

$$\delta J_v(u) = -\int_\Omega \delta u\big((\sqrt{-1}\partial\bar{\partial}u)^n - (\sqrt{-1}\partial\bar{\partial}v)^n\big), \tag{37}$$

and $J_v(v) = 0$.

Proof. Notice that $\mathcal{P}$ is connected, so we can connect v to $u \in \mathcal{P}$ by a path $u_t, 0 \leq t \leq 1$ such that $u_0 = v$ and $u_1 = u$. Define

$$J_v(u) = -\int_0^1 \int_\Omega \frac{\partial u_t}{\partial t}\big((\sqrt{-1}\partial\bar{\partial}u_t)^n - (\sqrt{-1}\partial\bar{\partial}v)^n\big)\, dt. \tag{38}$$

We need to show that the integral in (38) is independent of the choice of path u_t. Let $\delta u_t = w_t$ be a variation of the path. Then

$$w_1 = w_0 = 0 \qquad \text{and} \qquad w_t = 0 \quad \text{on } \partial\Omega,$$

and

$$\delta\int_0^1\int_\Omega \dot{u}\left((\sqrt{-1}\partial\bar{\partial}u)^n-(\sqrt{-1}\partial\bar{\partial}v)^n\right)dt$$
$$=\int_0^1\int_\Omega\Big(\dot{w}\big((\sqrt{-1}\partial\bar{\partial}u)^n-(\sqrt{-1}\partial\bar{\partial}v)^n\big)+\dot{u}\,n\sqrt{-1}\partial\bar{\partial}w(\sqrt{-1}\partial\bar{\partial}u)^{n-1}\Big)\,dt,$$

Since $w_0=w_1=0$, an integration by part with respect to t gives

$$\int_0^1\int_\Omega \dot{w}\big((\sqrt{-1}\partial\bar{\partial}u)^n-(\sqrt{-1}\partial\bar{\partial}v)^n\big)\,dt$$
$$=-\int_0^1\int_\Omega w\frac{d}{dt}(\sqrt{-1}\partial\bar{\partial}u)^n\,dt=-\int_0^1\int_\Omega\sqrt{-1}nw\partial\bar{\partial}\dot{u}(\sqrt{-1}\partial\bar{\partial}u)^{n-1}\,dt.$$

Notice that both w and $\dot{u}$ vanish on $\partial\Omega$, so an integration by part with respect to z gives

$$\begin{aligned}\int_\Omega\sqrt{-1}nw\partial\bar{\partial}\dot{u}(\sqrt{-1}\partial\bar{\partial}u)^{n-1}&=-\int_\Omega\sqrt{-1}n\partial w\wedge\bar{\partial}\dot{u}(\sqrt{-1}\partial\bar{\partial}u)^{n-1}\\&=\int_\Omega\sqrt{-1}n\dot{u}\partial\bar{\partial}w(\sqrt{-1}\partial\bar{\partial}u)^{n-1}.\end{aligned}$$

So

$$\delta\int_0^1\int_\Omega\dot{u}\left((\sqrt{-1}\partial\bar{\partial}u)^n-(\sqrt{-1}\partial\bar{\partial}v)^n\right)dt=0, \tag{39}$$

and the functional J is well defined. □

Using the J functional, we can define the F^0 functional as

$$F_v^0(u)=J_v(u)-\int_\Omega u(\sqrt{-1}\partial\bar{\partial}v)^n. \tag{40}$$

Then by Proposition 4, we have

$$\delta F_v^0(u)=-\int_\Omega\delta u(\sqrt{-1}\partial\bar{\partial}u)^n. \tag{41}$$

Proposition 5. *The basic properties of I, J and F^0 are following:*

(1) For any $u\in\mathcal{P}(\Omega,\varphi)$, $I_v(u)\ge J_v(u)\ge 0$.
(2) F^0 is convex on $\mathcal{P}(\Omega,\varphi)$, i.e. $\forall\,u_0,u_1\in\mathcal{P}$,

$$F^0\Big(\frac{u_0+u_1}{2}\Big)\le\frac{F^0(u_0)+F^0(u_1)}{2}. \tag{42}$$

(3) F^0 satisfies the cocycle condition*, i.e. $\forall\,u_1,u_2,u_3\in\mathcal{P}(\Omega,\varphi)$,*

$$F_{u_1}^0(u_2)+F_{u_2}^0(u_3)=F_{u_1}^0(u_3). \tag{43}$$

Proof. Let $w = (u - v)$ and $u_t = v + tw = (1-t)v + tu$, then

$$
\begin{aligned}
I_v(u) &= -\int_\Omega w\big((\sqrt{-1}\partial\bar\partial u)^n - (\sqrt{-1}\partial\bar\partial v)^n\big) \\
&= -\int_\Omega w\Big(\int_0^1 \frac{d}{dt}(\sqrt{-1}\partial\bar\partial u_t)^n\, dt\Big) \\
&= -\int_0^1 \int_\Omega \sqrt{-1}\, n w \partial\bar\partial w (\sqrt{-1}\partial\bar\partial u_t)^{n-1} \\
&= \int_0^1 \int_\Omega \sqrt{-1}\, n \partial w \wedge \bar\partial w \wedge (\sqrt{-1}\partial\bar\partial u_t)^{n-1} \geq 0,
\end{aligned} \tag{44}
$$

and

$$
\begin{aligned}
J_v(u) &= -\int_0^1 \int_\Omega w\big((\sqrt{-1}\partial\bar\partial u_t)^n - (\sqrt{-1}\partial\bar\partial v)^n\big)\, dt \\
&= -\int_0^1 \int_\Omega w\Big(\int_0^t \frac{d}{ds}(\sqrt{-1}\partial\bar\partial u_s)^n\, ds\Big)\, dt \\
&= -\int_0^1 \int_\Omega \int_0^t \sqrt{-1}\, n w \partial\bar\partial w (\sqrt{-1}\partial\bar\partial u_s)^{n-1}\, ds\, dt \\
&= \int_0^1 \int_\Omega (1-s)\sqrt{-1}\, n \partial w \wedge \bar\partial w \wedge (\sqrt{-1}\partial\bar\partial u_s)^{n-1}\, ds \geq 0.
\end{aligned} \tag{45}
$$

Compare (44) and (45), it is easy to see that

$$I_v(u) \geq J_v(u) \geq 0.$$

To prove (42), let $u_t = (1-t)u_0 + tu_1$, then

$$
\begin{aligned}
F^0(u_{1/2}) - F^0(u_0) &= -\int_0^{\frac{1}{2}} \int_\Omega (u_1 - u_0)\, (\sqrt{-1}\partial\bar\partial u_t)^n\, dt, \\
F^0(u_1) - F^0(u_{1/2}) &= -\int_{\frac{1}{2}}^1 \int_\Omega (u_1 - u_0)\, (\sqrt{-1}\partial\bar\partial u_t)^n\, dt.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} \int_\Omega (u_1 - u_0)\, (\sqrt{-1}\partial\bar\partial u_t)^n\, dt - \int_{\frac{1}{2}}^1 \int_\Omega (u_1 - u_0)\, (\sqrt{-1}\partial\bar\partial u_t)^n\, dt. \\
=&\int_0^{\frac{1}{2}} \int_\Omega (u_1 - u_0)\big((\sqrt{-1}\partial\bar\partial u_t)^n - (\sqrt{-1}\partial\bar\partial u_{t+1/2})^n\big)\, dt \\
=&2\int_0^{\frac{1}{2}} \int_\Omega (u_{t+1/2} - u_t)\big((\sqrt{-1}\partial\bar\partial u_t)^n - (\sqrt{-1}\partial\bar\partial u_{t+1/2})^n\big)\, dt \geq 0.
\end{aligned}
$$

So

$$F^0(u_1) - F^0(u_{1/2}) \geq F^0(u_{1/2}) - F^0(u_0).$$

The cocycle condition is a simple consequence of the variation formula 41. ∎

4. The Convergence

In this section, let us assume that both f and φ are independent of t. For $u \in \mathcal{P}(\Omega, \varphi)$, define

$$F(u) = F^0(u) + \int_\Omega G(z,u) dV, \tag{46}$$

where dV is the volume element in $\mathbb{C}^n$, and $G(z,s)$ is the function given by

$$G(z,s) = \int_0^s e^{-f(z,t)}\, dt.$$

Then the variation of F is

$$\delta F(u) = -\int_\Omega \delta u \big(\det(u_{\alpha\bar\beta}) - e^{-f(z,u)}\big)\, dV. \tag{47}$$

Proof of Theorem 2. We will follow Phong and Sturm's proof of the convergence of the Kähler-Ricci flow in [PS06]. For any $t > 0$, the function $u(\cdot, t)$ is in $\mathcal{P}(\Omega, \varphi)$. So by (47)

$$\begin{aligned} \frac{d}{dt} F(u) &= -\int_\Omega \dot u \big(\det(u_{\alpha\bar\beta}) - e^{-f(z,u)}\big) \\ &= -\int_\Omega \big(\log\det(u_{\alpha\bar\beta}) - (-f(z,u))\big)\big(\det(u_{\alpha\bar\beta}) - e^{-f(z,u)}\big) \le 0. \end{aligned}$$

Thus $F(u(\cdot,t))$ is monotonic decreasing as t approaches $+\infty$. On the other hand, $u(\cdot,t)$ is uniformly bounded in $\mathcal{C}^2(\overline{\Omega})$ by (10), so both $F^0(u(\cdot,t))$ and $f(z, u(\cdot,t))$ are uniformly bounded, hence $F(u)$ is bounded. Therefore

$$\int_0^\infty \int_\Omega \big(\log\det(u_{\alpha\bar\beta}) + f(z,u)\big)\big(\det(u_{\alpha\bar\beta}) - e^{-f(z,u)}\big)\, dt < \infty. \tag{48}$$

Observed that by the Mean Value Theorem, for $x, y \in \mathbb{R}$,

$$(x+y)(e^x - e^{-y}) = (x+y)^2 e^\eta \ge e^{\min(x,-y)}(x-y)^2,$$

where η is between x and $-y$. Thus

$$\big(\log\det(u_{\alpha\bar\beta}) + f\big)\big(\det(u_{\alpha\bar\beta}) - e^{-f}\big) \ge C_9\big(\log\det(u_{\alpha\bar\beta}) + f\big)^2 = C_9|\dot u|^2$$

where C_9 is independent of t. Hence

$$\int_0^\infty \|\dot u\|^2_{L^2(\Omega)}\, dt \le \infty \tag{49}$$

Let

$$Y(t) = \int_\Omega |\dot u(\cdot,t)|^2 \, \det(u_{\alpha\bar\beta})\, dV, \tag{50}$$

then

$$\dot Y = \int_\Omega \big(2\ddot u \dot u + \dot u^2 u^{\alpha\bar\beta} \dot u_{\alpha\bar\beta}\big) \det(u_{\alpha\bar\beta})\, dV.$$

Differentiate (1) in t,

$$\ddot u - u^{\alpha\bar\beta}\dot u_{\alpha\bar\beta} = f_u \dot u, \tag{51}$$

so

$$\begin{aligned}\dot{Y} &= \int_\Omega \big(2\dot{u}\dot{u}_{\alpha\bar{\beta}}u^{\alpha\bar{\beta}} + \dot{u}^2\big(2f_u + \ddot{u} - f_u\dot{u}\big)\big)\det(u_{\alpha\bar{\beta}})\,dV \\ &= \int_\Omega \big(\dot{u}^2\big(2f_u + \ddot{u} - f_u\dot{u}\big) - 2\dot{u}_\alpha\dot{u}_{\bar{\beta}}u^{\alpha\bar{\beta}}\big)\det(u_{\alpha\bar{\beta}})\,dV\end{aligned}$$

From (51), we get

$$\dddot{u} - u^{\alpha\bar{\beta}}\ddot{u}_{\alpha\bar{\beta}} - f_u\ddot{u} \leq f_{uu}\dot{u}^2$$

Since $f_u \leq 0$ and $f_{uu} \leq 0$, so $\ddot{u}$ is bounded from above by the maximum principle. Therefore

$$\dot{Y} \leq C_{10}\int_\Omega \dot{u}^2\det(u_{\alpha\bar{\beta}})\,dV = C_{10}Y,$$

and

$$Y(t) \leq Y(s)e^{C_{10}(t-s)} \qquad \text{for } t > s, \tag{52}$$

where C_{10} is independent of t. By (49), (52) and the uniform boundedness of $\det(u_{\alpha\bar{\beta}})$, we get

$$\lim_{t\to\infty}\|u(\cdot,t)\|_{L^2(\Omega)} = 0.$$

Since Ω is bounded, the L^2 norm controls the L^1 norm, hence

$$\lim_{t\to\infty}\|u(\cdot,t)\|_{L^1(\Omega)} = 0.$$

Notice that by the Mean Value Theorem,

$$|e^x - 1| < e^{|x|}|x|$$

so

$$\int_\Omega |e^{\dot{u}} - 1|\,dV \leq e^{\sup|\dot{u}|}\int_\Omega |\dot{u}|\,dV$$

Hence $e^{\dot{u}}$ converges to 1 in $L^1(\Omega)$ as t approaches $+\infty$. Now $u(\cdot,t)$ is bounded in $\mathcal{C}^2(\overline{\Omega})$, so $u(\cdot,t)$ converges to a unique function $\tilde{u}$, at least sequentially in $\mathcal{C}^1(\overline{\Omega})$, hence $f(z,u) \to f(z,\tilde{u})$ and

$$\det(\tilde{u}_{\alpha\bar{\beta}}) = \lim_{t\to\infty}\det(u(\cdot,t)_{\alpha\bar{\beta}}) = \lim_{t\to\infty}e^{\dot{u}-f(z,u)} = e^{-f(z,\tilde{u})},$$

i.e. $\tilde{u}$ solves (8).

□

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